Green's Identity :-

Green's 2nd identity :-

$$\iiint_R (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV = \iint_{\partial R} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) ds .$$

Green's 1st identity :-

$$\iiint_R \nabla\phi \, \nabla\psi \, dV = \iint_{\partial R} \phi \frac{\partial \psi}{\partial n}\, ds - \iiint_R \phi \nabla^2 \psi \, dV$$

If $\phi = \psi$, then..

$$\iiint_R (\nabla\phi)^2 dV = \iint_{\partial R} \phi \frac{\partial \phi}{\partial n}\, ds - \iiint_R \phi \nabla^2 \phi \, dV$$

Properties of Harmonic functions :-

Solutions of Laplace equation are called Harmonic functions which possess a number of interesting properties and they are presented in the following theorems.

I. If a harmonic function vanish everywhere on the boundary then it is identically zero everywhere.

$\Rightarrow$ Let ϕ be a harmonic function, then $\nabla^2\phi = 0$ in R.

Also, if $\phi = 0$ on ∂R, we shall show that $\phi = 0$ in $\bar{R} = R \cup \partial R$.

Recalling Green's 1st identity

$$\iiint_R (\nabla\phi)^2 dV = \iint_{\partial R} \phi \frac{\partial \phi}{\partial n}\, ds - \iiint_R \phi \nabla^2 \phi \, dV$$

$$\Rightarrow \iiint_R (\nabla\phi)^2 dV = 0$$

since $(\nabla\phi)^2$ is +ve, it follows that the integral will be satisfied only if $\nabla\phi = 0 \Rightarrow \phi$ is constant in R.

since ϕ is continuous in $\bar{R}$ and ϕ is zero on ∂R,

It follows that $\phi = 0$ in R.

II. If ϕ is a harmonic function in R and $\partial\phi/\partial n = 0$ then ϕ is constant in $\bar{R}$.

By Green's first identity

$$\iiint_R (\nabla\phi)^2 \, dV = \iint_{\partial R} \phi \frac{\partial\phi}{\partial n} \, ds - \iiint_R \phi \nabla^2\phi \, dV$$

$$\Rightarrow \iiint_R (\nabla\phi)^2 \, dV = 0$$

$$\Rightarrow \nabla\phi = 0$$

$\therefore$ ϕ is constant in R. Since the value of ϕ is not known on the boundary ∂R while $\partial\phi/\partial n = 0$. it is implied that ϕ is constant of ∂R.

$\therefore$ ϕ is constant in $\bar{R}$.

III. If the Dirichlet problem for a bounded region has a solution, then it is unique.

Let R be a bounded region and ∂R be its boundary.

The Dirichlet problem in R will be of the form:

$$\nabla^2\phi = 0 \text{ in } R \quad \text{and} \quad \phi = f \text{ on } \partial R \quad \text{where } f \in C^{(0)}$$

If ϕ_1 and ϕ_2 are two solutions of the interior Dirichlet problem, then

$$\nabla^2\phi_1 = 0 \text{ in } R, \quad \phi_1 = f \text{ on } \partial R$$

$$\nabla^2\phi_2 = 0 \text{ in } R \quad \phi_2 = f \text{ on } \partial R$$

Let $\psi = \phi_1 - \phi_2$, then

$$\nabla^2\psi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0 \text{ in } R$$

$$\psi = \phi_1 - \phi_2 = f - f = 0 \text{ on } \partial R$$

$\therefore$ $\nabla^2\psi = 0$ in R, $\psi = 0$ on ∂R.

By theorem - I, we obtain $\psi = 0$ on $\bar{R}$

$$\Rightarrow \phi_1 = \phi_2$$

Hence the solution of the Dirichlet problem is unique.

- Solve the boundary value problem described by..

PDE : $u_{tt} - c^2 u_{xx} = 0 \qquad 0 \le x \le l, \; t \ge 0$

BCs : $u(0,t) = u(l,t) = 0 \; . \qquad t \ge 0$

ICs : $u(x,0) = 10 \sin \frac{\pi x}{l} \; , \qquad 0 \le x \le l$

$u_t(x,0) = 0$

$\Rightarrow \quad u_{tt} = c^2 u_{xx} \; \ldots \text{(i)}$

Let the solution be. $u(x,t) = X(x).T(t)$, then..

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k$$

$\therefore \; X'' - kX = 0$

$T'' - kc^2 T = 0$

<u>case-I</u> :- when $k = 0$

$\therefore \; X'' = 0 \Rightarrow X = c_1 x + c_2$

$T'' = 0 \Rightarrow T = c_3 t + c_4$

$\therefore \; u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

using BC. $u(0,t) = u(l,t) = 0$, we get..

$c_1 = c_2 = 0$

$\therefore \; u(x,t) = 0$

which is a trivial solution, but for a non-trivial solution. we reject case I.

<u>case-II</u> :- when $k > 0 (= \lambda^2)$

$\therefore \; X'' - \lambda^2 X = 0 \Rightarrow X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$T'' - \lambda^2 c^2 T = 0 \Rightarrow T = c_3 e^{\lambda c t} + c_4 e^{-\lambda c t}$

$\therefore \; u(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{\lambda c t} + c_4 e^{-\lambda c t})$

using BC. $u(0,t) = u(l,t) = 0$. we get.

$c_1 + c_2 = 0$

$c_1 e^{\lambda l} + c_2 e^{-\lambda l} = 0 \qquad \Rightarrow \; c_1 = c_2 = 0$

$\therefore \; u(x,t) = 0$

which is a trivial solution. so we reject case-II.

-III :-

$k < 0 \; (= -\lambda^2)$

$X'' + \lambda^2 X = 0 \Rightarrow X = c_1 \cos\lambda x + c_2 \sin\lambda x$

$T'' + \lambda^2 c^2 T = 0 \Rightarrow T = c_3 \cos\lambda c t + c_4 \sin\lambda c t$

$\therefore u(x,t) = (c_1 \cos\lambda x + c_2 \sin\lambda x)(c_3 \cos\lambda c t + c_4 \sin\lambda c t)$

Now using B C $u(0,t) = u(l,t) = 0, \; t \geq 0$ we get ..

$$c_1 = 0$$

$$c_2 \sin\lambda l = 0$$

$\Rightarrow \sin\lambda l = 0 = \sin n\pi$ (for non-trivial solution $c_2 \neq 0$)

$$\Rightarrow \lambda = \frac{n\pi}{l}$$

$$\therefore u(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi c t}{l} + c_4 \sin\frac{n\pi c t}{l}\right)$$

$\therefore$ By the principal of superposition we get ..

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(A_n \cos\frac{n\pi c t}{l} + B_n \sin\frac{n\pi c t}{l}\right)$$

$$\therefore u_t(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[A_n\left(-\frac{n\pi c}{l}\right)\sin\frac{n\pi c t}{l} + B_n\left(\frac{n\pi c}{l}\right)\cos\frac{n\pi c t}{l}\right]$$

$u(x,0) = 10 \sin\frac{\pi x}{l}$

$$\therefore 10 \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

$n = 1$, we get $A_1 = 10$. while all other A_n's are zero.

$u_t(x,0) = 0$

$\sum B_n \frac{n\pi c}{l} \sin\frac{n\pi x}{l} = 0 \Rightarrow B_n = 0. \quad n = 1, 2 \ldots$

the required solution is ....

$$\cancel{u(x,t) = 10 \sin\frac{n\pi x}{l} \, a \cos\frac{n\pi c t}{l}}$$

$$u(x,t) = 10 \sin\frac{\pi x}{l} \cos\frac{\pi c t}{l}$$

• Solve the one-dimensional wave equation.

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x \le \pi, \; t \ge 0$$

subject to.

$u = 0$ when $x = 0$ and $x = \pi$

$u_t = 0$ when $t = 0$ and $u(x,0) = x$, $0 < x < \pi$.

$\Rightarrow$ $u_{tt} = c^2 u_{xx} \ldots$ (i)

Let the solution be ... $u(x,t) = X(x).T(t)$. then..

$$\frac{X''}{X} = \frac{1}{c^2}.\frac{T''}{T} = k$$

$\therefore X'' - kX = 0$

$T'' - c^2 kT = 0$

<u>case-I:-</u> when $k = 0$

$\therefore X'' = 0 \Rightarrow X = c_1 x + c_2$

$T'' = 0 \Rightarrow T = c_3 t + c_4$

$\therefore u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

Now, using BC. $u(0,t) = 0$, $u(\pi,t) = 0$. we get..

$c_1 = c_2 = 0$

$\therefore u(x,t) = 0$

which is a trivial solution, but for a non-trivial solution we reject case-I.

<u>case-II:-</u> when $k > 0 \; (= \lambda^2)$

$\therefore X'' - \lambda^2 X = 0 \Rightarrow X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$T'' - c^2\lambda^2 T = 0 \Rightarrow T = c_3 e^{\lambda c t} + c_4 e^{-\lambda c t}$

$\therefore u(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{\lambda c t} + c_4 e^{-\lambda c t})$

using B.C. $u(0,t) = u(\pi,t) = 0$. we get---.

$c_1 + c_2 = 0$

$c_1 e^{\lambda\pi} + c_2 e^{-\lambda\pi} = 0$ $\Rightarrow c_1 = c_2 = 0$

$\therefore u(x,t) = 0$

which is a trivial solution. so we reject case-II.

<u>case-III:-</u> when $k < 0 \; (= -\lambda^2)$

$\therefore X'' + \lambda^2 X = 0 \Rightarrow X = c_1 \cos\lambda x + c_2 \sin\lambda x$

$T'' + \lambda^2 c^2 T = 0 \Rightarrow T = c_3 \cos\lambda c t + c_4 \sin\lambda c t$

$u(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)(c_3 \cos \lambda ct + c_4 \sin \lambda ct)$

using BC $u(0,t) = u(\pi,t) = 0$ we get -

$$c_1 = 0$$

$$c_2 \sin \lambda \pi = 0$$

$$\Rightarrow \sin \lambda \pi = 0 = \sin n\pi \quad (\text{for non-trivial solution } c_2 \neq 0)$$

$$\Rightarrow \lambda = n$$

$$u(x,t) = c_2 \sin nx (c_3 \cos nct + c_4 \sin nct)$$

By the principal of superposition we get -

$$u(x,t) = \sum_{n=1}^{\infty} \sin nx (A_n \cos nct + B_n \sin nct)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin nx \left[A_n(-nc)\sin nct + B_n (nc)\cos nct\right]$$

$u_t(x,0) = 0$ gives

$$\sum_{n=1}^{\infty} B_n \, nc \sin nx = 0 \Rightarrow B_n = 0, \quad n = 1, 2 \ldots$$

$u(x,0) = x$ gives

$$x = \sum_{n=1}^{\infty} A_n \sin nx$$

where $A_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$

$$= \frac{2}{\pi}\left\{\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} + \int_0^{\pi} \frac{1}{n}\cos nx \, dx\right\}$$

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi}$$

$$= -\frac{2}{n}\cos n\pi$$

$$= \frac{2}{n}(-1)^{n+1}$$

~~The required~~ solution is .....

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx \cos nct$$

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- Solve... $u_{tt} = c^2 u_{xx}$. $0 \le x \le l,\ t > 0$

subject to... $u(0,t) = 0,\quad u(l,t) = 0 \quad \forall\, t$

$u(x,0) = 0 \quad u_t(x,0) = b \sin^3 \frac{\pi x}{l}$

$\Rightarrow \quad u_{tt} = c^2 u_{xx} \quad \ldots \text{(i)}$

Let the solution be... $u(x,t) = X(x).T(t)$, then..

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k$$

$\therefore \quad X'' - kX = 0$

$T'' - kc^2 T = 0$

<u>case-I</u>:- when $k = 0$

$\therefore X'' = 0 \Rightarrow X = c_1 x + c_2$

$T'' = 0 \Rightarrow T = c_3 t + c_4$

$\therefore u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

Now, using BC... $u(0,t) = u(l,t) = 0$ we get..

$c_1 = c_2 = 0$

$\therefore u(x,t) = 0$

which is a trivial solution, but for a non-trivial solution, we reject case-I.

<u>case-II</u>:- when $k > 0\ (= \lambda^2)$

$\therefore X'' - \lambda^2 X = 0 \Rightarrow X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$T'' - \lambda^2 c^2 T = 0 \Rightarrow T = c_3 e^{\lambda c t} + c_4 e^{-\lambda c t}$.

$\therefore u(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{\lambda c t} + c_4 e^{-\lambda c t})$

using BC... $u(0,t) = u(l,t) = 0$, we get..

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{\lambda l} + c_2 e^{-\lambda l} &= 0 \end{aligned}\right\} \Rightarrow c_1 = c_2 = 0$$

$\therefore u(x,t) = 0$

which is a trivial solution so we reject case-II.

<u>case-III</u>: when $k < 0\ (= -\lambda^2)$

$\therefore X'' + \lambda^2 X = 0 \Rightarrow X = c_1 \cos \lambda x + c_2 \sin \lambda x$

$T'' + \lambda^2 c^2 T = 0 \Rightarrow T = c_3 \cos \lambda c t + c_4 \sin \lambda c t$

$u(x,t) = (c_1 \cos\lambda x + c_2 \sin\lambda x)(c_3 \cos\lambda ct + c_4 \sin\lambda ct)$

using BC $u(0,t) = u(l,t) = 0$ we get.

$c_1 = 0$

$c_2 \sin\lambda l = 0$

$\Rightarrow \sin\lambda l = 0 = \sin n\pi$ (for non-trivial solution $c_2 \neq 0$)

$\Rightarrow \lambda = \frac{n\pi}{l}$

$$u(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right)$$

By the principal of superposition we get...

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[\left(-\frac{n\pi c}{l}\right). A_n \sin\frac{n\pi ct}{l} + B_n.\left(\frac{n\pi c}{l}\right)\cos\frac{n\pi ct}{l}\right]$$

$u(x,0) = 0$ gives...

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = 0 \Rightarrow A_n = 0,\quad n = 1, 2.$$

$u_t(x,0) = b \sin^3\frac{\pi x}{l}$ gives..

$$\sum_{n=1}^{\infty} B_n \frac{n\pi c}{l} \sin\frac{n\pi x}{l} = b\sin^3\frac{\pi x}{l} = b\left(\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right)$$

$$\cancel{\sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} = b\left(\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right)}$$

putting $n = 1$, and $n = 3$ we get..

$\cancel{c_1 = \frac{3b}{4},\ c_3 = -\frac{b}{4}}$ $B_1 = \frac{3bl}{4\pi c}$, $B_3 = -\frac{1b}{12\pi c}$

other ~~An's are zero~~. B_n's are zero..

The required solution is.....

$$u(x,t) = \sum \frac{3bl}{4\pi c}\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \frac{bl}{12\pi c}\sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}$$

$$u(x,t) = \frac{bl}{12c\pi}\left[9\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}\right]$$

• Determine the solution of the one-dimensional wave equation :-

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0, \quad 0<x<a,\ t>0$$

with c as a constant under the following initial and boundary conditions:

(i) $\phi(x,0) = f(x) = \begin{cases} x/b & 0 \le x \le b \\ (a-x)/(a-b), & b < x \le a \end{cases}$

(ii) $\frac{\partial \phi}{\partial t}(x,0) = 0, \quad 0<x<a$

(iii) $\phi(0,t) = \phi(a,t) = 0, \quad t \geq 0$

$\Rightarrow \quad \phi_{tt} - c^2 \phi_{xx} = 0$ .... (i)

let the solution be ... $\phi(x,t) = X(x).T(t)$, then ....

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k$$

$\therefore \ X'' - kX = 0$

$T'' - kc^2 T = 0$

<u>case - I</u> :- when $k=0$

$\therefore \ X'' = 0 \Rightarrow X = c_1 x + c_2$

$T'' = 0 \Rightarrow T = c_3 t + c_4$

$\therefore \ \phi(x,t) = (c_1 x + c_2)(c_3 t + c_4)$

using BC.... $\phi(0,t) = \phi(a,t) = 0$, we get....

$c_1 = c_2 = 0$

$\therefore \ \phi(x,t) = 0$

which is a trivial solution, but for a non-trivial solution we reject case - I

<u>case - II</u> :- when $k>0 \ (=\lambda^2)$.

$\therefore \ X'' - \lambda^2 X = 0 \Rightarrow X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$T'' - \lambda^2 c^2 T = 0 \Rightarrow T = c_3 e^{\lambda c t} + c_4 e^{-\lambda c t}$

$\therefore \ \phi(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{\lambda c t} + c_4 e^{-\lambda c t})$

using BC ... $\phi(0,t) = \phi(a,t) = 0$, we get ...

$c_1 + c_2 = 0$

$c_1 e^{\lambda a} + c_2 e^{-\lambda a} = 0 \quad \Rightarrow c_1 = c_2 = 0$

$\therefore \ \phi(x,t) = 0$

which is a trivial solution, so we reject case - II case-III :-

when $\lambda < 0$ $(= -\lambda^2)$

$\therefore$ $X'' + \lambda^2 X = 0 \Rightarrow X = c_1 \cos \lambda x + c_2 \sin \lambda x$

$T'' + \lambda^2 c^2 T = 0 \Rightarrow T = c_3 \cos \lambda c t + c_4 \sin \lambda c t$

$\therefore$ $\phi(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)(c_3 \cos \lambda c t + c_4 \sin \lambda c t)$

Now using BC.. $\phi(0,t) = \phi(a,t) = 0$ we get..

$c_1 = 0$

$c_2 \sin \lambda a = 0$

$\Rightarrow \sin \lambda a = 0 = \sin n\pi$ (for non-trivial solution $c_2 \neq 0$)

$\Rightarrow \lambda = \frac{n\pi}{a}$

$\therefore$ $\phi(x,t) = c_2 \sin \frac{n\pi x}{a} \left(c_3 \cos \frac{n\pi c t}{a} + c_4 \sin \frac{n\pi c t}{a} \right)$

$\therefore$ By the principal of superposition....

$$\phi(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \left(A_n \cos \frac{n\pi c t}{a} + B_n \sin \frac{n\pi c t}{a} \right)$$

$$\therefore \phi_t(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \left[A_n \left(-\frac{n\pi c}{a}\right) \sin \frac{n\pi c t}{a} + B_n \left(\frac{n\pi c}{a}\right) \cos \frac{n\pi c t}{a} \right]$$

Now $\phi_t(x,0) = 0$ gives...

$$\sum_{n=1}^{\infty} B_n \cdot \frac{n\pi c}{a} \cdot \sin \frac{n\pi x}{a} = 0 \Rightarrow B_n = 0, \quad n = 1, 2 \ldots$$

Again. $\phi(x,0) = f(x)$

$$\therefore f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a}$$

$$\therefore A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx$$

$$= \frac{2}{a} \int_0^b \frac{x}{b} \sin \frac{n\pi x}{a}\, dx + \frac{2}{a} \int_b^a \frac{a-x}{a-b} \sin \frac{n\pi x}{a}\, dx$$

$$= \cancel{\frac{2}{a}\left\{\left[\frac{1}{b} x \left(\frac{-\cos(n\pi x/a)}{n\pi/a}\right)\right]_0^b - \frac{1}{b}\right.}$$

$$= \frac{2}{ab} \int_0^b x \sin \frac{n\pi x}{a}\, dx + \frac{2}{a(a-b)} \int_b^a (a-x) \sin \frac{n\pi x}{a}\, dx$$

$$= \frac{2}{ab}\left[-x\frac{\cos\frac{n\pi x}{a}}{n\pi/a} + \frac{\sin\frac{n\pi x}{a}}{n^2\pi^2/a^2}\right]_0^b + \frac{2}{a(a-b)}\left[-a\frac{\cos\frac{n\pi x}{a}}{n\pi/a} + x\frac{\cos\frac{n\pi x}{a}}{n\pi/a} - \frac{\sin\frac{n\pi x}{a}}{n^2\pi^2/a^2}\right]_b^a$$

$$= \cancel{\frac{2}{ab}\left[-\frac{ab}{n\pi} + \frac{a^2}{n^2\pi^2}\sin\frac{n\pi b}{a}\right] + \frac{2}{a(a-b)}}\Big[$$

$$= \frac{2}{ab}\left[-\frac{ab}{n\pi}\cos\frac{n\pi b}{a} + \frac{a^2}{n^2\pi^2}\sin\frac{n\pi b}{a}\right] + \frac{2}{a(a-b)}\left[-\frac{a^2}{n\pi}\cos n\pi + \frac{a^2}{n\pi}\cos n\pi - \frac{a^2}{n^2\pi^2}\sin n\pi + \frac{a^2}{n\pi}\cos\frac{n\pi b}{a} - \frac{ab}{n\pi}\cos\frac{n\pi b}{a} + \frac{a^2}{n^2\pi^2}\sin\frac{n\pi b}{a}\right]$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi b}{a} + \frac{2a}{bn^2\pi^2}\sin\frac{n\pi b}{a} + \cancel{2a}\,\frac{2}{n\pi}\cos\frac{n\pi b}{a} + \frac{2a}{(a-b)n^2\pi^2}\sin\frac{n\pi b}{a}$$

$$= \left(\frac{2a}{b} + \frac{2a}{a-b}\right)\frac{1}{n^2\pi^2}\sin\frac{n\pi b}{a}$$

$$= \frac{2a^2}{n^2\pi^2 b(a-b)}\sin\frac{n\pi b}{a}$$

Hence the required solution is —

$$\phi(x,t) = \sum_{n=1}^{\infty}\frac{2a^2}{n^2\pi^2 b(a-b)}\sin\frac{n\pi b}{a}\sin\frac{n\pi x}{a}\cos\frac{n\pi ct}{a}$$

• Properties of Harmonic functions:-

16 IV. If the Neumann problem for a bounded region has a solution, then it is either unique or it differs ~~for~~ from one another by a constant only.

⇒ Let ~~ϕ_1 and ϕ_2 be two distinct solutions of~~ us consider a neumann problem of the form.

$$\left.\begin{aligned} \nabla^2\phi &= 0 \quad \text{in } R \\ \frac{\partial\phi}{\partial n} &= f \text{ on } \partial R \end{aligned}\right\} \quad \text{---- (i)}$$

where ∂R is the boundary of the region R and $f \in C^{(0)}$ in R

Let ϕ_1 and ϕ_2 be two solutions of (i)

$$\therefore \nabla^2\phi_1 = 0 \text{ in } R \text{ and } \frac{\partial\phi_1}{\partial n} = f \text{ on } \partial R$$

$$\nabla^2\phi_2 = 0 \text{ in } R \text{ and } \frac{\partial\phi_2}{\partial n} = f \text{ on } \partial R$$

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$\Psi = \phi_1 - \phi_2$ then.

$$\nabla^2 \Psi = \nabla^2 \phi_1 - \nabla^2 \phi_2 = 0 \quad \text{in } R \quad \cdots \text{(ii)}$$

$$\frac{\partial \Psi}{\partial n} = \frac{\partial \phi_1}{\partial n} - \frac{\partial \phi_2}{\partial n} = 0 \quad \text{on } \partial R \quad \cdots \text{(iii)}$$

From equation (iii) we get

$$\Psi = \text{constant on } \bar{R} = R \cup \partial R$$

$$\Rightarrow \phi_1 - \phi_2 = \text{constant}$$

Thus the solution of the Neumann problem is not unique.

The solution of a Neumann problem can differ from one another by a constant only.

Solution of Laplace equation :-

1. Dirichlet problem for a rectangle :-

The Dirichlet problem for a rectangle is defined by.—

$$\nabla^2 \psi = 0 \qquad 0 \le x \le a, \quad 0 \le y \le b$$

subject to. $\psi(x, b) = \psi(a, y) = 0, \quad \psi(0, y) = 0, \quad \psi(x, 0) = f(x)$

The PDE is .. $\nabla^2 \Psi = 0$

$$\Rightarrow \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \quad \cdots \text{(i)}$$

Let the solution is the form ...

$$\Psi(x, y) = X(x) \cdot Y(y)$$

putting this in (i) we get.

$$\frac{X''}{X} = -\frac{Y''}{Y} = k \qquad (k = \text{constant})$$

or

$$X'' - kX = 0$$

$$Y'' + kY = 0$$

Now three cases may arise.

Case - I : when $K > 0 \; (= P^2)$, then.

$$X'' - P^2 X = 0 \Rightarrow X = c_1 e^{Px} + c_2 e^{-Px}$$

$$Y'' + P^2 Y = 0 \Rightarrow Y = c_3 \cos Py + c_4 \sin Py$$

$$\therefore \; \psi(x, y) = (c_1 e^{Px} + c_2 e^{-Px})(c_3 \cos Py + c_4 \sin Py)$$

Now, using BC. $\psi(a, y) = 0$ and $\psi(0, y) = 0$ we get..

$$c_1 e^{Pa} + c_2 e^{-Pa} = 0$$

$$c_1 + c_2 = 0 \qquad \Rightarrow c_1 = c_2 = 0$$

$$\therefore \; \psi(x, y) = 0$$

which is a trivial solution. but for a non-trivial solution, we reject case - I.

Case - II :- when $K = 0$, then

$$X'' = 0 \Rightarrow X = c_1 x + c_2$$

$$Y'' = 0 \Rightarrow Y = c_3 y + c_4$$

$$\therefore \; \psi(x, y) = (c_1 x + c_2)(c_3 y + c_4)$$

Now, using BC. $\psi(a, y) = \psi(0, y) = 0$, we get..

$$c_1 a + c_2 = 0$$

$$c_2 = 0 \qquad \Rightarrow c_1 = c_2 = 0$$

$$\therefore \; \psi(x, y) = 0$$

which is also a trivial solution, so we reject case - II

Case - III :- when $K < 0 \; (= -P^2)$. then

$$X'' + P^2 X = 0 \Rightarrow X = c_1 \cos Px + c_2 \sin Px$$

$$Y'' - P^2 Y = 0 \Rightarrow Y = c_3 e^{Py} + c_4 e^{-Py}$$

$$\therefore \; \psi(x, y) = (c_1 \cos Px + c_2 \sin Px)(c_3 e^{Py} + c_4 e^{-Py})$$

Now, using BC $\psi(0, y) = \psi(a, y) = 0$, we get..

$$c_1 = 0$$

$$c_2 \sin Pa = 0$$

$\Rightarrow \sin Pa = 0 = \sin n\pi$ (for a non-trivial solution $c_2 \neq 0$)

$$p = \frac{n\pi}{a}$$

$$\psi(x, y) = c_2 \sin\frac{n\pi x}{a}\left(c_3 e^{n\pi y/a} + c_4 e^{-n\pi y/a}\right)$$

$\therefore$ By the principal of superposition we get.

$$\psi(x, y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{a}\left(a_n e^{n\pi y/a} + b_n e^{-n\pi y/a}\right) \quad \text{-- (ii)}$$

Now using BC. $\psi(x, b) = 0$. we get..

$$\sin\frac{n\pi x}{a}\left(a_n e^{n\pi b/a} + b_n e^{-n\pi b/a}\right) = 0$$

$$\Rightarrow b_n = -a_n \frac{e^{n\pi b/a}}{e^{-n\pi b/a}}, \quad n = 1, 2 \ldots$$

Thus equation (ii) becomes.

$$\psi(x, y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{a}\, a_n \left(e^{n\pi y/a} - \frac{e^{-n\pi y/a}}{e^{-n\pi b/a}} \cdot e^{n\pi b/a}\right)$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{e^{-n\pi b/a}} \sin\frac{n\pi x}{a}\left(e^{n\pi(y-b)/a} - e^{-n\pi(y-b)/a}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2a_n}{e^{-n\pi b/a}} \sin\frac{n\pi x}{a} \sinh\frac{n\pi(y-b)}{a}$$

$$= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi(y-b)}{a}$$

Now using BC $\psi(x, 0) = f(x)$. we get..

$$f(x) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{-n\pi b}{a}\right) \sin\frac{n\pi x}{a}$$

$$\therefore A_n \sinh\left(\frac{-n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$$

$$\therefore A_n = \frac{2}{a} \frac{1}{\sinh\left(\frac{-n\pi b}{a}\right)} \int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$$

$\therefore$ The required solution of the ~~gr~~ Dirichlet problem is given by..

$$\psi(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi(y-b)}{a}$$

where $$A_n = \frac{2}{a}\frac{1}{\sinh\left(-\frac{n\pi b}{a}\right)} \int_0^a f(x) \sin\frac{n\pi x}{a}\, dx$$

- Neumann problem for a rectangle :-

The Neumann problem for a rectangle is given by.

$$\nabla^2\psi = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$

subject to.. $\psi_x(0,y) = \psi_x(a,y) = 0$, $\psi_y(x,0) = 0$, $\psi_y(x,b) = f(x)$

The PDE is $\nabla^2\psi = 0$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \quad \text{--- (i)}$$

Let the solution is the form..

$$\psi(x,y) = X(x).Y(y)$$

putting this in (i) we get...

$$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

$$\therefore X'' - kX = 0$$
$$Y'' + kY = 0$$

Now three cases may arise.

case - I :- when $k > 0 \ (= p^2)$, then...

$$X'' - p^2 X = 0 \Rightarrow X = c_1 e^{px} + c_2 e^{-px}$$

$$Y'' + p^2 Y = 0 \Rightarrow Y = c_3 \cos py + c_4 \sin py$$

$$\therefore \psi(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

$$\psi_x(x,y) = p(c_1 e^{px} - c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

Now using B.C. $\psi_x(0,y) = \psi_x(a,y) = 0$ we get.

$$\left.\begin{aligned} c_1 - c_2 &= 0 \\ c_1 e^{pa} - c_2 e^{-pa} &= 0 \end{aligned}\right\} \Rightarrow c_1 = c_2 = 0$$

$\therefore \psi(x,y) = 0$

which is a trivial solution, but for a non-trivial solution we reject case-I.

Case-II :- when $k=0$ then.

$$X'' = 0 \Rightarrow X = c_1 x + c_2$$

$$Y'' = 0 \Rightarrow Y = c_3 y + c_4$$

$$\therefore \psi(x,y) = (c_1 x + c_2)(c_3 y + c_4)$$

$$\therefore \psi_x(x,y) = c_1 (c_3 y + c_4)$$

$$\psi_y(x,y) = c_3 (c_1 x + c_2)$$

Now, using B.C $\psi_x(0,y) = 0$ we get. $c_1 = 0$

and $\psi_y(x,0) = 0$, we get $c_3 = 0$

$$\therefore \psi(x,y) = c_2 c_4 = A_0$$

Case-III :- when $k<0\ (=-p^2)$, then..

$$X'' + p^2 X = 0 \Rightarrow X = c_1 \cos px + c_2 \sin px$$

$$Y'' - p^2 Y = 0 \Rightarrow Y = c_3 e^{py} + c_4 e^{-py}$$

$$\therefore \psi(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

Now, ~~using~~ $\psi_x(x,y) = -p(c_1 \sin px - c_2 \cos px)(c_3 e^{py} + c_4 e^{-py})$

using BC $\psi_x(0,y) = \psi_x(a,y) = 0$, we get...

$$c_2 = 0$$

$$c_1 \sin pa = 0 \qquad (\text{for non-trivial solution } c_1 \neq 0)$$

$$\Rightarrow \sin pa = 0 = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{a}, \quad n = \cancel{1,2}\ 0,1,2\ldots$$

$$\psi(x,y) = c_2 \overset{\cos}{\cancel{\sin}} \frac{n\pi x}{a} \left(c_3 e^{n\pi y/a} + c_4 e^{-n\pi y/a}\right)$$

$\therefore$ By the principal of superposition we get.

$$\psi(x,y) = \sum_{n=0}^{\infty} \cos \frac{n\pi x}{a} \left(a_n e^{n\pi y/a} + b_n e^{-n\pi y/a}\right) \quad \ldots\ (ii)$$

$$\therefore \quad \psi_y(x,y) = \sum_{n=0}^{\infty} \cos\frac{n\pi x}{a} \ \frac{n\pi}{a}\left(a_n e^{n\pi y/a} - b_n e^{-n\pi y/a}\right).$$

using BC. $\psi_y(x,0) = 0$ we get.

$$a_n - b_n = 0 \Rightarrow a_n = b_n$$

$\therefore$ Thus equation (ii) becomes..

$$\psi(x,y) = \sum_{n=0}^{\infty} \cos\frac{n\pi x}{a} \ 2a_n\left(\frac{e^{n\pi y/a} + e^{-n\pi y/a}}{2}\right)$$

$$= \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{a} \cosh\frac{n\pi y}{a}$$

$$\therefore \quad \psi_y(x,y) = \sum_{n=1}^{\infty} A_n . \frac{n\pi}{a} \cos\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}.$$

using BC $\psi_y(x,b) = f(x)$. we get..

$$f(x) = \sum_{n=1}^{\infty} A_n . \frac{n\pi}{a} \sinh\frac{n\pi b}{a} \cos\frac{n\pi x}{a}$$

$$\therefore \quad A_n . \frac{n\pi}{a} \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x) \cos\frac{n\pi x}{a}\, dx$$

$$\Rightarrow A_n = \frac{2}{n\pi}\frac{1}{\sinh\frac{n\pi b}{a}}\int_0^a f(x) \cos\frac{n\pi x}{a}\, dx$$

$\therefore$ The required solution of the Neumann problem is given by...

$$\psi(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a} \cosh\frac{n\pi y}{a}$$

$$\text{where} \quad A_n = \frac{2}{n\pi}\frac{1}{\sinh\frac{n\pi b}{a}}\int_0^a f(x) \cos\frac{n\pi x}{a}\, dx$$

- Solve the Problem....

$$\nabla^2 u = 0$$

subject to.. $\quad u(0,y) = 0. \quad u(a,y) = 0$

$$u(x,b) = 0 \quad . \quad u(x,0) = Tx(x-a)$$

The PDE is $\nabla^2 u = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (i)}$$

Let the solution be ..

$$u(x,y) = X(x) \cdot Y(y)$$

putting this in (i) we get ..

$$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

$\therefore \; X'' - kX = 0$

$Y'' + kY = 0$

Now three cases may arise

Case - I :- when $k > 0 \; (= p^2)$. then

$X'' - p^2 X = 0 \Rightarrow X = c_1 e^{px} + c_2 e^{-px}$

$Y'' + p^2 Y = 0 \Rightarrow Y = c_3 \cos py + c_4 \sin py$

$\therefore \; u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

Now using B.C $u(0,y) = u(a,y) = 0$ we get ..

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{pa} + c_2 e^{-pa} &= 0 \end{aligned}\right\} \Rightarrow c_1 = c_2 = 0$$

$\therefore \; u(x,y) = 0$

which is a trivial solution, but for a non-trivial solution, we reject case - I.

Case - II :- when $k = 0$, then.

$X'' = 0 \Rightarrow X = c_1 x + c_2$

$Y'' = 0 \Rightarrow Y = c_3 y + c_4$

$\therefore \; u(x,y) = (c_1 x + c_2)(c_3 y + c_4)$

Now using BC $u(0,y) = u(a,y) = 0$ we get ..

$c_2 = 0$

$c_1 a + c_2 = 0 \Rightarrow c_1 = 0$

$\therefore \; u(x,y) = 0$

which is also a ~~tri~~ trivial solution so we reject case - II.

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Case - III :- when $k < 0 \, (= -p^2)$, then..

$$X'' + p^2 X = 0 \Rightarrow X = c_1 \cos px + c_2 \sin px$$

$$Y'' - p^2 Y = 0 \Rightarrow Y = c_3 e^{py} + c_4 e^{-py}$$

$$\therefore u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

Now, using BC $u(0,y) = u(a,y) = 0$, we get..

$$c_1 = 0$$

$$c_2 \sin pa = 0$$

$$\Rightarrow \sin pa = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{a}, \quad n = 1, 2 \ldots$$

$$\therefore u(x,y) = c_2 \sin \frac{n\pi x}{a} \left(c_3 e^{n\pi y/a} + c_4 e^{-n\pi y/a}\right)$$

By the principal of superposition we get..

$$u(x,y) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \left(a_n e^{n\pi y/a} + b_n e^{-n\pi y/a}\right)$$

Now, using BC. $u(x,b) = 0$ we get..

$$\sin \frac{n\pi x}{a} \left(a_n e^{n\pi b/a} + b_n e^{-n\pi b/a}\right) = 0$$

$$\Rightarrow b_n = -a_n \frac{e^{n\pi b/a}}{e^{-n\pi b/a}}$$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \cdot 2a_n \left(\frac{e^{n\pi(y-b)/a} - e^{-n\pi(y-b)/a}}{2 \cdot e^{-n\pi b/a}}\right)$$

$$= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(y-b)}{a}$$

Now using BC.. $u(x,0) = Tx(x-a)$ we get..

$$Tx(x-a) = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{n\pi b}{a}\right) \cdot \sin \frac{n\pi x}{a}$$

$$\therefore A_n \sinh\left(-\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a Tx(x-a) \sin \frac{n\pi x}{a} \, dx$$

$$\Rightarrow A_n \frac{a}{2T} \sinh\left(-\frac{n\pi b}{a}\right) = \int_0^a (x^2 - ax) \sin \frac{n\pi x}{a} \, dx$$

$$\Rightarrow A_n \frac{a}{2T} \sinh\left(-\frac{n\pi b}{a}\right) = \int_0^a x^2 \sin\frac{n\pi x}{a}\, dx - a\int_0^a x \sin\frac{n\pi x}{a}\, dx$$

$$= \left[x^2\left(-\frac{\cos n\pi x/a}{n\pi/a}\right)\right]_0^a + \int_0^a 2x \frac{\cos n\pi x/a}{n\pi/a}\, dx - a\left[x\left(-\frac{\cos n\pi x/a}{n\pi/a}\right)\right]_0^a$$

$$+ \int_0^a \frac{\cos n\pi x/a}{n\pi/a}\, dx\Big]$$

$$= \left[x^2\left(-\frac{\cos n\pi x/a}{n\pi/a}\right)\right]_0^a + 2\left\{x \frac{\sin n\pi x/a}{n^2\pi^2/a^2}\Big|_0^a - \int_0^a \frac{\sin n\pi x/a}{n^2\pi^2/a^2}\, dx\right\}$$

$$-a\left[-x\cos\frac{n\pi x/a}{n\pi/a} + \frac{\sin n\pi x/a}{n^2\pi^2/a^2}\right]_0^a$$

$$= \left[-x^2 \frac{\cos n\pi x/a}{n\pi/a} + 2x\frac{\sin n\pi x/a}{n^2\pi^2/a^2} + 2\frac{\cos n\pi x/a}{n^3\pi^3/a^3} + ax\cos\frac{n\pi x/a}{n\pi/a}\right.$$

$$\left. - a\sin\frac{n\pi x/a}{n^2\pi^2/a^2}\right]_0^a$$

$$= 2\frac{a^3}{n^3\pi^3}\left[(-1)^n - 1\right]$$

~~= 0, when n is even~~

=

$$\therefore A_n = \operatorname{cosech}\left(-\frac{n\pi b}{a}\right)\frac{4a^2T}{n^3\pi^3}\left[(-1)^n - 1\right]$$

Hence the required general solution is given by...

$$u(x,y) = \sum_{n=1}^{\infty} \operatorname{cosech}\left(-\frac{n\pi b}{a}\right)\frac{4Ta^2}{n^3\pi^3}\left[(-1)^n - 1\right]\sin\frac{n\pi x}{a}\sinh\frac{n\pi(y-b)}{a}$$

- Show that if the two-dimensional Laplace equation $\nabla^2 u = 0$ is transformed by introducing plane polar co-ordinates r, θ defined by the relations $x = r\cos\theta$, $y = r\sin\theta$, it takes the form.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$\Rightarrow \quad x = r\cos\theta \qquad y = r\sin\theta$

$r^2 = x^2+y^2, \qquad \theta = \tan^{-1} y/x$

$r_x = \cos\theta \qquad r_y = \sin\theta$

$$\theta_x = \frac{1}{1+y^2/x^2}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

$$\theta_y = \frac{1}{1+y^2/x^2}\cdot\left(1/x\right) = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

since $u = u(r,\theta)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= u_r\cos\theta - u_\theta\frac{\sin\theta}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial r}\left(u_r\cos\theta - u_\theta\frac{\sin\theta}{r}\right)\cdot\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\left(u_r\cos\theta - u_\theta\frac{\sin\theta}{r}\right)\cdot\frac{\partial\theta}{\partial x}$$

$$= \left(u_{rr}\cos\theta - u_{\theta r}\frac{\sin\theta}{r} + u_\theta\frac{\sin\theta}{r^2}\right)\cos\theta + \left(u_{r\theta}\cos\theta - u_r\sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta\frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}$$

$$= u_r\sin\theta + u_\theta\frac{\cos\theta}{r}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial r}\left(u_r\sin\theta + u_\theta\frac{\cos\theta}{r}\right)\frac{\partial r}{\partial y} + \frac{\partial}{\partial\theta}\left(u_r\sin\theta + u_\theta\frac{\cos\theta}{r}\right)\cdot\frac{\partial\theta}{\partial y}$$

$$= \left(u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r} - u_\theta\frac{\cos\theta}{r^2}\right)\sin\theta + \left(u_{r\theta}\sin\theta + u_r\cos\theta + u_{\theta\theta}\frac{\cos\theta}{r} - u_\theta\frac{\sin\theta}{r}\right)\left(\frac{\cos\theta}{r}\right)$$

putting this in $\nabla^2 u = 0$ we get --

$$u_{rr}\cos^2\theta - u_{\theta r}\frac{\cancel{\sin\theta\cos\theta}}{r} + u_\theta\frac{\cancel{\sin\theta\cos\theta}}{r^2} - u_{r\theta}\frac{\cancel{\cos\theta\sin\theta}}{r} + u_r\frac{\sin^2\theta}{r}$$

$$+ u_{\theta\theta}\frac{\sin^2\theta}{r^2} + u_\theta\frac{\cancel{\cos\theta\sin\theta}}{r^2} + u_{rr}\sin^2\theta + u_{r\theta}\frac{\cancel{\cos\theta\sin\theta}}{r} - u_\theta\frac{\cancel{\cos\theta\sin\theta}}{r^2}$$

$$+ u_{r\theta}\frac{\cancel{\sin\theta\cos\theta}}{r} + u_r\frac{\cos^2\theta}{r} + u_{\theta\theta}\frac{\cos^2\theta}{r^2} - u_\theta\frac{\cancel{\sin\theta\cos\theta}}{r^2} = 0$$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Which is the laplace's equation in polar co-ordinates.

Show that in cylindrical co-ordinates r, θ, z defined by the relations $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ the laplace equation $\nabla^2 u = 0$ takes the form..

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

The laplace equation is $\nabla^2 u = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{--- (i)}$$

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}(y/x)$

$r_x = \cos\theta, \quad r_y = \sin\theta, \quad \theta_x = -\frac{\sin\theta}{r}, \quad \theta_y = \frac{\cos\theta}{r}$

$u = u(r, \theta, z)$

$$u_x = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$= u_r \cos\theta - u_\theta \frac{\sin\theta}{r}$$

$$u_{xx} = \frac{\partial}{\partial r}\left(u_r\cos\theta - u_\theta\frac{\sin\theta}{r}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(u_r\cos\theta - u_\theta\frac{\sin\theta}{r}\right)\cdot\frac{\partial \theta}{\partial x}$$

$$+ \frac{\partial}{\partial z}\left(u_r\cos\theta - u_\theta\frac{\sin\theta}{r}\right)\cdot\frac{\partial z}{\partial x}$$

$$= \left(u_{rr}\cos\theta - u_{\theta r}\frac{\sin\theta}{r} + u_\theta\frac{\sin\theta}{r^2}\right)\cos\theta + \left(u_{r\theta}\cos\theta - u_r\sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta\frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right)$$

$$u_y = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}$$

$$= u_r \sin\theta + u_\theta\frac{\cos\theta}{r}$$

$\Rightarrow \quad x = r\cos\theta \qquad y = r\sin\theta$

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1} y/x$$

$$\therefore \quad r_x = \cos\theta \qquad r_y = \sin\theta$$

$$\theta_x = \frac{1}{1 + y^2/x^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

$$\theta_y = \frac{1}{1 + y^2/x^2} \cdot \left(1/x\right) = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$$

since $u = u(r, \theta)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$= u_r \cos\theta - u_\theta \frac{\sin\theta}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial r}\left(u_r\cos\theta - u_\theta \frac{\sin\theta}{r}\right) \cdot \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(u_r\cos\theta - u_\theta \frac{\sin\theta}{r}\right) \cdot \frac{\partial \theta}{\partial x}$$

$$= \left(u_{rr}\cos\theta - u_{\theta r}\frac{\sin\theta}{r} + u_\theta \frac{\sin\theta}{r^2}\right)\cos\theta + \left(u_{r\theta}\cos\theta - u_r \sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta \frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$= u_r \sin\theta + u_\theta \frac{\cos\theta}{r}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial r}\left(u_r\sin\theta + u_\theta \frac{\cos\theta}{r}\right) \frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\left(u_r\sin\theta + u_\theta \frac{\cos\theta}{r}\right) \cdot \frac{\partial \theta}{\partial y}$$

$$= \left(u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r} - u_\theta \frac{\cos\theta}{r^2}\right)\sin\theta + \left(u_{r\theta}\sin\theta + u_r \cos\theta + u_{\theta\theta}\frac{\cos\theta}{r} - u_\theta \frac{\sin\theta}{r}\right)\left(\frac{\cos\theta}{r}\right)$$

putting this in $\nabla^2 u = 0$ we get --

$$u_{rr}\cos^2\theta - u_{\theta r}\cancel{\frac{\sin\theta\cos\theta}{r}} + u_\theta \cancel{\frac{\sin\theta\cos\theta}{r^2}} - u_{r\theta}\cancel{\frac{\cos\theta\sin\theta}{r}} + u_r \frac{\sin^2\theta}{r}$$

$$+ u_{\theta\theta}\frac{\sin^2\theta}{r^2} + u_\theta \cancel{\frac{\cos\theta\sin\theta}{r^2}} + u_{rr}\sin^2\theta + u_{r\theta}\cancel{\frac{\cos\theta\sin\theta}{r}} - u_\theta \cancel{\frac{\cos\theta\sin\theta}{r^2}}$$

$$+ u_{r\theta}\cancel{\frac{\sin\theta\cos\theta}{r}} + u_r \frac{\cos^2\theta}{r} + u_{\theta\theta}\frac{\cos^2\theta}{r^2} - u_\theta \cancel{\frac{\sin\theta\cos\theta}{r^2}} = 0$$

$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

which is the laplace's equation in polar co-ordinates.

Show that in cylindrical co-ordinates r, θ, z defined by the relations $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ the laplace equation $\nabla^2 u = 0$ takes the form..

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

The laplace equation is $\nabla^2 u = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \text{ --- (i)}$$

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$r^2 = x^2 + y^2$, $\theta = \tan^{-1}(y/x)$

$r_x = \cos\theta$, $r_y = \sin\theta$, $\theta_x = -\frac{\sin\theta}{r}$, $\theta_y = \frac{\cos\theta}{r}$

$u = u(r, \theta, z)$

$$u_x = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$= u_r \cos\theta - u_\theta \frac{\sin\theta}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial r}\left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right) \cdot \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right) \cdot \frac{\partial \theta}{\partial x}$$

$$+ \frac{\partial}{\partial z}\left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right) \cdot \frac{\partial z}{\partial x}$$

$$= \left(u_{rr} \cos\theta - u_{\theta r} \frac{\sin\theta}{r} + u_\theta \frac{\sin\theta}{r^2}\right) \cos\theta + \left(u_{r\theta} \cos\theta - u_r \sin\theta - u_{\theta\theta} \frac{\sin\theta}{r} - u_\theta \frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y}$$

$$= u_r \sin\theta + u_\theta \frac{\cos\theta}{r}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial r}\left(u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right)\cdot\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\left(u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right)\frac{\partial \theta}{\partial y}$$

$$+ \frac{\partial}{\partial z}\left(u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right)\cdot\frac{\partial z}{\partial y}$$

$$= \left(u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r} - u_\theta \frac{\cos\theta}{r^2}\right)\sin\theta + \left(u_{r\theta}\sin\theta + u_r\cos\theta\right.$$

$$\left. + u_{\theta\theta}\frac{\cos\theta}{r} - u_\theta\frac{\sin\theta}{r}\right)\left(\frac{\cos\theta}{r}\right)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial z}$$

$$= \frac{\partial u}{\partial z}$$

$$\therefore \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial z^2}$$

$\therefore$ putting this values in (i) we get -

$$\cancel{\frac{\partial^2 u}{\partial r^2}}$$

$$u_{rr}\cos^2\theta - u_{\theta r}\cancel{\frac{\sin\theta\cos\theta}{r}} + u_\theta\cancel{\frac{\sin\theta\cos\theta}{r^2}} - u_{r\theta}\cancel{\frac{\sin\theta\cos\theta}{r}} + \frac{u_r}{r}\sin^2\theta$$

$$+ \frac{u_{\theta\theta}}{r^2}\sin^2\theta + u_\theta\cancel{\frac{\cos\theta\sin\theta}{r^2}} + u_{rr}\sin^2\theta + u_{r\theta}\cancel{\frac{\cos\theta\sin\theta}{r}} - u_\theta\cancel{\frac{\cos\theta\sin\theta}{r^2}}$$

$$+ u_{r\theta}\cancel{\frac{\sin\theta\cos\theta}{r}} + \frac{u_r}{r}\cos^2\theta + \frac{u_{\theta\theta}}{r^2}\cos^2\theta - u_\theta\cancel{\frac{\sin\theta\cos\theta}{r^2}} + u_{zz} = 0$$

$$\Rightarrow u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

which is the Laplace equation in cylindrical co-ordinates.

- **Interior Dirichlet problem for a circle :-**

The Dirichlet problem for a circle is defined by :

$$\nabla^2 u = 0 \qquad 0 \le r \le a, \quad 0 \le \theta \le 2\pi$$

$$\text{B.C.: } u(a,\theta) = f(\theta), \qquad 0 \le \theta \le 2\pi$$

Now the equation $\nabla^2 u = 0$ in polar co-ordinates can be written as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \ldots (i)$$

... the solution be $u(r,\theta) = R(r)\,H(\theta)$

putting this in (i) we get ..

$$R''H + \frac{1}{r}R'H + \frac{1}{r^2}RH'' = 0$$

$$\Rightarrow \frac{r^2R'' + rR'}{R} = -\frac{H''}{H} = k$$

$\Rightarrow r^2R'' + rR' - kR = 0$

$H'' + kH = 0$

Case - I :- Let $k = \lambda^2$, then ..

$$r^2R'' + rR' - \lambda^2 R = 0$$

which is a Euler type of equation and can be solved by putting $r = e^z$.

The solution is. $R = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$

$$\Rightarrow R = c_1 r^{\lambda} + c_2 r^{-\lambda}$$

$H'' + \lambda^2 H = 0 \Rightarrow H = c_3 \cos\lambda\theta + c_4 \sin\lambda\theta$

$\therefore u(r,\theta) = (c_1 r^{\lambda} + c_2 r^{-\lambda})(c_3\cos\lambda\theta + c_4\sin\lambda\theta)$ ----- (ii)

Case - II :- Let $k = -\lambda^2$, then

$$r^2R'' + rR' + \lambda^2R = 0 \quad \text{and} \quad H'' - \lambda^2 H = 0$$

The solutions are ...

$$R = c_1\cos(\lambda \ln r) + c_2\sin(\lambda \ln r)$$

$$H = c_3 e^{\lambda\theta} + c_4 e^{-\lambda\theta}$$

$\therefore u(r,\theta) = (c_1\cos(\lambda \ln r) + c_2\sin(\lambda\ln r))(c_3e^{\lambda\theta} + c_4e^{-\lambda\theta})$ ----- (iii)

Case - III :- Let $k = 0$, then....

$$r^2R'' + rR' = 0$$

Put $R' = V$ then

$$r\frac{dV}{dr} + V = 0$$

$$\frac{dV}{V} + \frac{dr}{r} = 0$$

integrating $\ln Vr = \ln c_1$

$$V = \frac{c_1}{r} = \frac{dR}{dr}$$